ON SOME INTEGRAL RELATIONSHIPS FOR COMMENSURATE TRANSMISSION-LINE NETWORKS

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Abstract

Several integral relationships are presented for commensurate transmission-line networks. The integrals focus on the face that $Z(1)$ for such net works, where $Z(S)$ is the input immittance of the network, is associated with a real or redundant unit element prefacing the network Three bandwidth restrictions are derived Some applications of the integral relationships are presented.

For commensurate transmission-line networks it is convenient to use Richards[1] variable S , where

$$S=\tanh\left\{\frac{\tau s}{2}\right\}=\Sigma + i\,\Omega,$$

$\tau = 2l/v$, the round trip delay for the shortest commensurate length line,

l = length of the shortest commensurate length line,

v = velocity of propagation,

$s = \sigma + i\omega$, the complex frequency variable of lumped element networks.

Richards proved that driving point immittances (impedances or admittances) $Z(S)$ are rational functions of S and are positive real. In this paper we consider several integral relationships for general immittance functions $Z(S)$ expressable in the form,

$$Z(S) = F(S) + M(S)$$

$$F(S) = \text{Foster pre-amble} = A_1^{\infty} S + \frac{A_{-1}^{O}}{S} + \sum_{k=1}^{L} \frac{2A_{-1}^{k}\, S}{S^2 + \Omega_k^2}$$

$$M(S) = \frac{a_0 + a_1 S + a_2 S^2 + \ldots a_n S^n}{b_0 + b_1 S + b_2 S^2 + \ldots b_m S^m}\text{ , with } n = m \text{ or } m-1.$$

Also, at infinity, $M(S)$ can be expanded into

$$\lim_{S\to\infty} M(S) = m_{\infty} + \frac{m_{-1}}{S} + \frac{m_{-2}}{S^2} + \ldots$$

The integral of primary interest is

$$\oint_C \frac{Z(S)}{S^2 - 1}\, dS\ , \tag{1}$$

where C is the Bromwich[2] contour consisting of the $\Sigma = 0$ axis, and the infinite semi-circle enclosing the RHP. Details of the evaluating Eq. (1) are given in the expanded paper. The final result is

$$Z(1) = \frac{2}{\pi}\int_0^{\infty} \frac{R(\Omega)\, d\Omega}{\Omega^2 + 1} + A_1^{\infty} + A_{-1}^{O} + 2\sum_{k=1}^{L} \frac{A_{-1}^{k}}{1 + \Omega_k^2}, \tag{2}$$

where $R(\Omega) + i\,X(\Omega) = Z(i\Omega)$. On the "real-frequency axis" $S = i\Omega = i\tan\theta$, where $\theta = \omega l/v$ is the electrical length Substitution into Eq. (2) results in

$$Z(1) = \frac{2}{\pi}\int_0^{\pi/2} R(\theta)\, d\theta + A_1^{\infty} + A_{-1}^{O} + 2\sum_{k=1}^{L} \frac{A_{-1}^{k}}{1 + \Omega_k^2}\ . \tag{3}$$

The integral on the RHS of Eq (3) is the average of $R(\theta)$ over $\pi/2$ radians. Hence transposing, Eq (3) states

$$R_{avg.} = Z(1) - 2\sum_{k=1}^{L} \frac{A_{-1}^{k}}{1+\Omega_k^2} - A_{-1}^{O} - A_1^{\infty} \tag{4}$$

Thus, *the average value of the real part of $Z(S)$ over $\pi/2$ radians equals $Z(1)$ less the weighted values of the residues of its Foster preamble* Equations (2) – (4) are particularly useful forms since $Z(1)$ can be interpreted as the characteristic immittance of a unit element[3] prefacing the Z Network.

WEAK-LIMIT BANDWIDTH LAWS

Equations (3) and (4) may be used to derive a "weak-limit" bandwidth law applicable to all networks having a positive real input immittance, and a less general weak-limit bandwidth law applicable to a restricted class of networks These laws will be discussed in the presentation but are merely summarized here·

1. $$\frac{2}{\pi}\int_0^{\pi/2} P_{in}/P_{avail}\, d\theta \leqslant M/R_1 \left\{Z(1) - A_1^{\infty} - A_{-1}^{O} - 2\sum_{k=1}^{L} \frac{A_{-1}^{k}}{(1+\Omega_k^2)}\right\} \tag{5}$$

with a maximum uncertainty of $U(M)$.

2. $$\frac{2}{\pi}\int_0^{\pi/2} P_{avail}/P_{in}\, d\theta \geqslant \frac{[1 + Z(1)/R_1]^2}{4Z(1)/R_1} \tag{6}$$

Equation (5) is general, while Eq. (6) holds only for networks where the input immittance has non-zero real part, and no poles or zeros on the real frequency axis. In Eqs (5) and (6) R_1 is the source immittance, M is a positive parameter $\leqslant 4$, and $U(M)$ is a monotonic decreasing function of M with a maximum of 1 In the expanded paper, a procedure is given for determining M so as to minimize the RHS of Eq. (5) We define the RHS of Eq. (5) as the *weak-limit.*

RETURN LOSS BANDWIDTH LAW

Using the previous results, the average return loss may be computed. We consider the integral

$$\oint_C \frac{\ln \Gamma(S)}{S^2 - 1}\, dS\ ,$$

where $\Gamma(S)$ is the reflection coefficient of a given network N The reflection coefficient may be expressed

$$\Gamma(S) = A\,\frac{\prod_i (S - z_i)}{\prod_k (S - p_k)}$$

where A is real, z_i, p_k are in general complex, Re $(p_k) < 0$, but the Re (z_i) unrestricted. It can be shown that the same final result is obtained for negative A as positive A. We treat the latter case here Two separate cases are considered

Case I Re $(z_i) < 0$ for all i Then, by Eq. (4)

$$\frac{2}{\pi}\int_0^{\pi/2} \ln|\Gamma(\theta)|\,d\theta = \ln|\Gamma(1)|. \quad (7)$$

Case II Re $(z_i) > 0$ for $I \leq i \leq I'$ In this case, following Bode's procedure[4], we form

$$\tilde{\Gamma}(S) = \Gamma(S) \prod_i \frac{(S + z_i)}{(S - z_i)}. \quad (8)$$

Note that

$|\tilde{\Gamma}(S)| = |\Gamma(S)|$ for $S = i\Omega$ Therefore, by Eq. (4),

$$\frac{2}{\pi}\int_0^{\pi/2} \ln|\Gamma(\theta)|\,d\theta = \ln|\Gamma(1)| + \ln\left|\prod_{i=I}^{I'} \frac{1+z_i}{1-z_i}\right| \quad (9)$$

The first term on the RHS of Eq (9) is always less than zero, while the second term on the RHS of Eq (9) is always greater than zero. Thus, Cases I and II can conveniently be expressed in the single inequality.

$$\frac{2}{\pi}\int_0^{\pi/2} \ln\left|\frac{1}{\Gamma(\theta)}\right| d\theta \leq \ln\left|\frac{1}{\Gamma(1)}\right|. \quad (10)$$

Equation (10) states

the average return loss in nepers over $\pi/2$ radians is less than or equal the return loss at $S = 1$.

It is interesting to note that if the first element of the network N is a unit element of characteristic impedance Z, the average return loss cannot exceed

$$\ln\left|\frac{Z+1}{Z-1}\right| \quad (11)$$

regardless of the remainder of N.

An important consequence can be drawn from these results regarding the bandwidth of cascaded, stepped-impedance transformers and directional couplers. For impedance transformers one desires to minimize Γ over the matching bandwidth. However, the average return loss cannot exceed Eq. (11) Thus, the gain-bandwidth performance of the transformer is limited (at least) by the impedance levels of the *first* and *last* quarter-wavelength lines.

For a cascaded directional coupler, Γ for the even-mode input impedance is in one-to-one correspondence with the coupling coefficient of the coupler. In this case, one wishes Γ to be constant over as wide a bandwidth as possible But, again, the average coupling $(\ln|\frac{1}{\Gamma}|)$ cannot exceed Eq (11) Thus, the bandwidth of the coupler is limited (at least) by the even-mode impedance of the *first* and *last* coupling sections. The above limits may be weak limits depending on the zeros of the reflection coefficient. But, in all cases they cannot be exceeded

MEASUREMENT APPLICATIONS

In addition to providing bandwidth information on Z, Eqs. (2) -- (4) can be utilized in certain measurement methods. For example, Eqs. (3) and (4) suggest a simple CW procedure for measuring the characteristic impedance of an unknown transmission line. Consider a transmission line of unknown characteristic impedance R_X terminated in an arbitrary real load R_L. At a suitable reference plane,

$$Z(S) = R_X \frac{R_L + R_X S}{R_X + R_L S} = M(S)$$

In this case $A_{-1}^{0} = A_1^{\infty} = A_{-1}^{k} = 0$. Therefore,

$$Z(1) = R_X = \frac{2}{\pi}\int_0^{\pi/2} R(\theta)\,d\theta = R_{avg}$$

Thus, a measurement of $R(\theta)$ averaged over $\pi/2$ radians equals the characteristic impedance of the line Note that knowledge of the value of R_L is not required An advantage of this CW approach is that averaging the measurement tends to reduce random measurement errors.

For a second example, consider a network consisting of a line (characteristic impedance R) terminated by a shunt stub (characteristic admittance C) in parallel with a real load R_L. Evaluation of Eq (4) for the average of R_{in} and G_{in} yields

$$\left\{R_{in}\right\}_{avg} = R/(1 + RC)$$

$$\left\{R_{in}\right\}_{avg}^{-1} = G + C, \quad (G = 1/R) \quad (12)$$

$$\left\{G_{in}\right\}_{avg} = G \quad (13)$$

A possible application of the latter results is to the experimental determination of self and mutual capacitance (and hence coupling) of some coupled-line geometries Consider the network shown in Fig. 1. The relationships between the equivalent circuit parameters (which correspond to the current example) and the coupled-line parameters are

$$v C_{11} = G + C \quad (14)$$

$$v C_{12} = \sqrt{C(G + C)} \quad (15)$$

Thus, CW measurements of $\left\{R_{in}\right\}_{avg}$ and $\left\{G_{in}\right\}_{avg.}$ together with Eqs (12) – (15), yield the coupled-line parameters

Numerous other examples are possible

CONCLUSIONS

Use of the weighting function $(S^2 - 1)^{-1}$ in network function contour integrals resulted in several network inter-relationships and 3 bandwidth constraints The integral relationships can be used in the CW determination of network parameters in certain cases

Some specific results were

1. For a network preceded by a unit element, the average real part of the input immittance is less than, or equal to, the characteristic immittance of the unit element.

2. In general, the average of the real part of the input immittance Z(S) is equal to Z(1) less the weighted residues of its Foster preamble

3. In general,

$$\frac{2}{\pi}\int_0^{\pi/2} P_{in}/P_{avail}\,d\theta \leq \text{the weak-limit (as defined in the text)}$$

4 For networks whose input immittance has non-zero real part and no poles or zeros on the $j\Omega$ axis,

$$\frac{2}{\pi}\int_{o}^{\pi/2} P_{avail}/P_{in}\, d\theta \geqslant \frac{[1+z(1)]^2}{4z(1)}$$

5 In general,

$$\frac{2}{\pi}\int_{o}^{\pi/2} \ln\left|\frac{1}{\Gamma(\theta)}\right| d\theta \leqslant \ln\left|\frac{1}{\Gamma(1)}\right|$$

REFERENCES

[1]P.I.Richards, "Resistor Transmission Line Circuits," Proc. IRE, vol. 36 pp 217-220, February 1948

[2]N.W McLachlan, "Complex Variable Theory and Transform Calculus," Chpt 4, 2nd ed., Chambridge 1963

[3]H. Ozaki, J Ishii, "Syhthesis of a Class of Strip-line Filters" IRE Transion Circuit Theory, CT-5, pp 104-109, June 1958

[4]H. Bode, "Network Analysis and Feedback Amplifier Design," Van Nostrand 1945

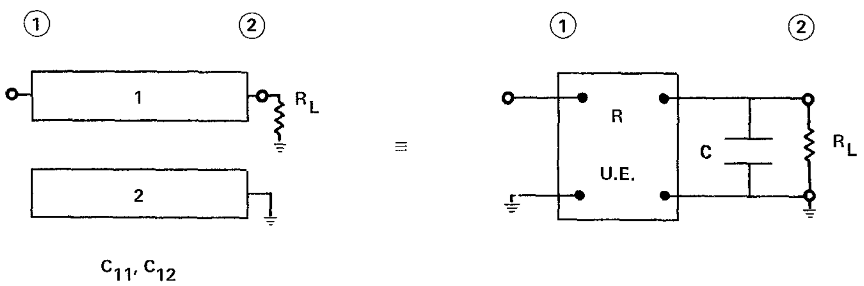

Fig. 1 Symmetrical Coupled-Line Geometry and Equivalent Circuit

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